Stochastic Rounding and its Probabilistic Backward Error Analysis

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Stochastic rounding and its probabilistic backward error analysis,
\( \text{fl}(x \text{ op} y) = (x \text{ op} y)(1 + \delta), \quad |\delta| \leq u, \text{ op} \in \{+, -, \times, \}/ \)
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**Lemma (Higham (2002))**

If \( |\delta_i| \leq u \) for \( i = 1 : n \), and \( nu < 1 \), then

\[
\prod_{i=1}^{n} (1 + \delta_i) = 1 + \theta_n, \quad |\theta_n| \leq \gamma_n,
\]

with

\[
\gamma_n := \frac{nu}{1 - nu} = nu + O(u^2).
\]
A probabilistic bound

\[ \tilde{\gamma}_n(\lambda) := \exp \left( \frac{\lambda \sqrt{nu + nu^2}}{1 - u} \right) - 1 = \lambda \sqrt{nu} + O(u^2). \]

**Theorem (Higham & Mary (2019))**

Let \( \delta_1, \delta_2, \ldots, \delta_n \) be independent random variables of mean zero with \( |\delta_i| \leq u, i = 1 : n \). Then for any \( \lambda > 0 \) we have

\[ \prod_{i=1}^{n} (1 + \delta_i) = 1 + \theta_n, \quad |\theta_n| \leq \tilde{\gamma}_n(\lambda) \]

which holds with probability at least

\[ P(\lambda) = 1 - 2 \exp(-\lambda^2/2). \]
Stochastic rounding

Given adjacent floating-point numbers $a, b$ and $x \in \mathbb{R}$ so that $a \leq x \leq b$, we have

$$\text{fl}(x) = \begin{cases} b \text{ with probability } p = (x - a)/(b - a), \\ a \text{ with probability } 1 - p. \end{cases}$$

- Called **Mode 1** stochastic rounding (SR).
- Gaining interest in machine learning.
Rounding errors

\[ \text{fl}(x \text{ op } y) = (x \text{ op } y)(1 + \delta), \quad |\delta| \leq 2u, \text{ op } \in \{+,-,\times,1\} \]
Theorem (C, Higham & Mary, 2021)

The rounding errors $\delta_1, \delta_2, \ldots, \delta_n$ produced by stochastic rounding are mean independent, mean zero random variables such that

$$E(\delta_k) = E(\delta_k | \delta_{k-1}, \ldots, \delta_1) = 0.$$
A new theorem

Theorem (C, Higham & Mary, 2021)

Let $\delta_1, \delta_2, \ldots, \delta_n$ be mean independent random variables of mean zero with $|\delta_i| \leq u$, $i = 1 : n$. Then for any $\lambda > 0$ we have

$$\prod_{i=1}^{n} (1 + \delta_i) = 1 + \theta_n, \quad |\theta_n| \leq \tilde{\gamma}_n(\lambda)$$

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- SR satisfies these assumptions (with the substitution \( u \leftarrow 2u \)).
- Rule of thumb becomes a rule!
Example: inner product

- Want to compute $y = a^T b$, $a, b \in \mathbb{R}^n$.

- When using SR, we have the backward error result:

  $$
  \hat{y} = (a + \Delta a)^T b, \\
  |\Delta a| \leq \tilde{\gamma}_n(\lambda)|a| \approx \lambda \sqrt{n u} |a|.
  $$

- The result holds with probability at least $1 - 2n \exp(-\lambda^2/2)$.

- Compare with the worst case bound for round to nearest (RTN)

  $$
  |\Delta a| \leq \gamma_n |a| \approx nu |a|.
  $$
Numerical experiments

- Compute inner product $y = a^T b$ for $a, b$ sampled uniformly from $[0, 1]$.

- Work in fp16 ($u = 2^{-11}$).

- Use the implementation of SR provided by chop (Higham and Pranesh, 2019).

- [https://github.com/higham/chop](https://github.com/higham/chop)
Numerical experiments

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Stochastic Rounding
As the intermediate value $y_i = y_{i-1} + a_i b_i$ grows, the spacing between nearby floating-point values increases.

We reach a point where under RTN the sum can no longer grow.

SR solves this issue by jumping in the “wrong” direction.
Conclusions

- SR produces mean independent, mean zero rounding errors.

- SR provides backward error bounds that are proportional to $\sqrt{nu}$.

- SR can prove much more accurate than RTN in certain scenarios.
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